

Cosmological Challenges in Theories with Extra Dimensions and Remarks on the Horizon Problem

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We consider the cosmology that results if our observable universe is a 3-brane in a higher dimensional universe. In particular, we focus on the case where our 3-brane is located at the Z_2 symmetry fixed plane of a Z_2 symmetric five-dimensional spacetime, as in the Hořava-Witten model compactified on a Calabi-Yau manifold. As our first result, we find that there can be substantial modifications to the standard Friedmann-Robertson-Walker (FRW) cosmology; as a consequence, a large class of such models is observationally inconsistent. In particular, any relationship between the Hubble constant and the energy density on our brane is possible, including (but not only) FRW. Generically, due to the existence of the bulk and the boundary conditions on the orbifold fixed plane, the relationship is not FRW, and hence cosmological constraints coming from big bang nucleosynthesis, structure formation, and the age of the universe difficult to satisfy. We do wish to point out, however, that some specific choices for the bulk stress-energy tensor components do reproduce normal FRW cosmology on our brane, and we have constructed an explicit example. As our second result, for a broad class of models, we find a somewhat surprising fact: the stabilization of the radius of the extra dimension and hence the four dimensional Planck mass requires unrealistic fine-tuning of the equation of state on our 3-brane. In the last third of the paper, we make remarks about causality and the horizon problem that apply to *any* theory in which the volume of the extra dimension determines the four-dimensional gravitational coupling. We point out that some of the assumptions that lead to the usual inflationary requirements are modified.

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I. INTRODUCTION

Recently, there has been a great deal of renewed interest in the physics of extra dimensions, the existence of which is a generic feature of string theories. In this paper, we consider the cosmology that results if our observable universe is a 3-brane in a higher dimensional universe. In particular, we focus on the case where our 3-brane is located at the Z_2 symmetry fixed plane of a Z_2 symmetric five-dimensional spacetime; our work applies to any 3-brane with this Z_2 symmetry. Hence this work applies to a class of models proposed by Hořava and Witten [1] which compactify M theory on an orbifold S^1/Z_2 , with one of the E_8 on one orbifold fixed plane and the other E_8 on the other orbifold fixed plane. The size of one of the extra dimensions is determined by the orbifold radius (we will call this extra dimension ‘the bulk’). In the Hořava and Witten models, once the rest of the extra dimensions are compactified on a Calabi-Yau manifold with a radius smaller than the orbifold radius, the standard model (SM) fields charged under a subgroup of E_8 are confined to one of the two 3-branes while gravity is free to propagate in the bulk. There has been some effort to explore the cosmologies resulting from such a class of models which we will refer to as the Hořava-Witten models. Some of this work has focused on the inflationary periods or on the modifications to the particle physics related phenomena as a result of Kaluza-Klein excitations. There has been additional study of cosmology in scenarios without a Z_2 symmetry. References of previous cosmological work in the context of our observable universe as a 3-brane in a higher dimensional context include [2–4].

In this work we focus on rather general cosmological considerations of a 3-brane with Z_2 symmetry. We explore consequences of the main novel features: i) the boundary conditions on the orbifold fixed planes induced by the Z_2 symmetry and ii) the dependence of the 4-dimensional Planck constant on the orbifold proper length. In the first two-thirds of this paper, we restrict ourselves to models in which the spacetime has the structure $\mathcal{M}_4 \times S^1/Z_2$ where \mathcal{M}_4 is foliated by flat, homogeneous and isotropic 3-slices and find that, generically, substantial modifications to the observable Friedmann-Robertson-Walker (FRW) universe result. In the last third of the paper, we make remarks about

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causality and the horizon problem that apply to *any* theory in which the volume of the extra dimension determines the four-dimensional gravitational coupling. While this work was in progress, some of the same results have been presented by [5]; however, much of our work covers additional aspects of cosmology in theories with extra dimensions as well¹.

Because the novel elements of Hořava-Witten models result in a different set of cosmological equations, observational constraints can be imposed on such models. We find that, from a five dimensional point of view in general, FRW cosmologies are not the generic solutions on the brane containing our world. In particular, there is no manifest dynamical mechanism which drives the theory to have the fields living on the brane dominating the stress energy. Hence, one rarely recovers the standard relationship $H^2 \sim \rho/m_{pl}^2$ where ρ is the energy density of the fields living on the brane and H is the Hubble speed on the 3-brane. The main point is that the behavior of H and thus the evolution of our observable universe is controlled by the moduli² from the extra dimensions and the attending boundary conditions of the spacetime, rather than by what is on our brane. This statement is true even when the bulk is “empty.” One can think of this as a version of the moduli problem, which states that our 3-brane’s stress energy can be generically subdominant compared to the effect of the moduli (particularly those not confined to our 3-brane) on our 3-brane’s cosmological history. It is possible although not generic to reproduce ordinary FRW on our brane, and we construct a specific example in which FRW does result. Although we discuss this problem in the context of a compact S^1/Z_2 extra dimension motivated by the Hořava-Witten model, this conclusion should apply to any theory in which our observable universe is a 3-brane located at the Z_2 symmetry fixed point of a Z_2 symmetric five dimensional spacetime.

The altered relationship between H and ρ leads to drastic modifications of the results from Big Bang Nucleosynthesis. The temperature/time relationship at the time of Big Bang Nucleosynthesis is drastically modified, and the element abundances produced in a generic model are in violation of observation. We also discuss the growth of density perturbations required for large scale structure formation and find that the standard results are no longer obtained: galaxy formation cannot proceed in the usual fashion.

We also find the result that, in a broad class of models, it is generically impossible to stabilize the radius of the orbifold without causing the equation of state $w = P/\rho$ on our brane to diverge; *i.e.*, fine tuning of the equation of state is required in order to allow one to fix the four dimensional Planck mass today at a constant value. Because the 3-brane observable universe that we live in has an energy density whose scale is much smaller than the fundamental Planck scale, we would naively expect that the stress energy of the 3-brane is irrelevant as far as the stabilization of the orbifold is concerned. However, this need not be true in general since the brane stress energy must be consistently coupled to the bulk stress energy. We give an example where an absolutely static orbifold is physically forbidden due to 1) this bulk/boundary coupling and 2) the fact that the equation of state on the boundary is constrained to satisfy $-1 < w = P/\rho < 1$. In this class of models, we find that without fine tuning of the equation of state to $w = -2/3$ or without having just a cosmological constant type of equation of state $w = -1$, it is impossible to “pin” the four dimensional Planck mass to today’s value; *i.e.*, it is impossible to stabilize the radius of the orbifold. Instead, the four dimensional Planck mass must still be changing, albeit slowly, in order to satisfy the constraints and the equations of motion. We find that even when the Newton’s constant is allowed to vary slowly (within experimental bounds), it is difficult to construct a realistic model that does not contain an unnaturally large number. The large number arises from the mismatch between the energy density on our 3-brane and the energy density set by the scale of the fundamental Planck’s constant, *i.e.*, a version of the cosmological constant problem. Although we have only shown this difficulty with stabilizing the radius of the extra dimension (and hence the four-dimensional Planck mass) explicitly for specific models, we suspect that this difficulty may persist more generally for other models with extra dimensions.

We also make some remarks about causality and the solution to the horizon problem in theories with extra dimensions. These remarks apply to any theory, regardless of boundary conditions, in which the observable universe lies on a three-brane and gravity is described by an extra dimension. We caution that many assumptions go into the standard inflationary requirements, and illustrate how a few of these assumptions may be modified here. In particular, the assumption that $H^2 \sim T_c^2/m_{pl}(t_c)$ at an early time may be modified. In addition, the fact that the value of the four dimensional Planck mass changes as the radius of compactification changes must not be forgotten in inflation models, and in fact may be exploited to solve the horizon problem (see also [8]).

In Section II, we present the model with a discussion of boundary conditions. In Section III, we discuss the resulting

¹Also, as this work was being submitted, we became aware of [6] and [7], which also discuss brane cosmology.

²In this paper, we use the word moduli to refer to any light field degrees of freedom that contributes to the effective stress energy tensor.

nonstandard cosmology on our brane. In Section IV, we explore constraints from nucleosynthesis, structure formation, and the age of the universe in the context of the nonstandard cosmology. In Section V we show that, in a broad class of models, the four dimensional Planck mass cannot be fixed at a constant value. In Section VI we discuss causality and the horizon problem. We conclude in Section VII.

II. MODEL

In the first part of this paper we restrict ourselves to a scenario in which standard model particles are confined to one of the two orbifold fixed planes (3-branes) and gravity resides in the bulk. We consider a space-time with the structure $S^1/Z_2 \times \mathcal{M}_4$ where S^1/Z_2 is an orbifold and \mathcal{M}_4 is a smooth manifold. We use the metric convention $+, -, -, -, -$. Let the action be decomposed as

$$S_5 = \frac{-1}{2\kappa_5^2} \int d^5x \sqrt{g} \mathcal{R} + S_{orb} + S_{boundary} + S_{GH} \quad (1)$$

where \mathcal{R} is the Ricci scalar in 5 dimensions, g is the absolute value of the determinant of the five dimensional metric g_{MN} , κ_5 is the five-dimensional Newton's constant ($\kappa_5^2 \equiv 1/m_{pl,5}^3$), S_{orb} represents the orbifold (or the bulk) action, $S_{boundary}$ represents the boundary (orbifold fixed plane) action (i.e. integrated only over the boundary), and S_{GH} is the Gibbons-Hawking boundary term to be discussed below.

A. Metric and Einstein Equations

We consider the metric of the form

$$ds^2 = e^{2\nu(\tau,u)} d\tau^2 - e^{2\alpha(\tau,u)} d\mathbf{x}^2 - e^{2\beta(\tau,u)} du^2 \quad (2)$$

which foliates the space into flat, homogeneous and isotropic spatial 3-planes. Here $\mathbf{x} = x_1, x_2, x_3$ are the coordinates on the spatial 3-planes while u is the orbifold coordinate (this metric was considered for example by Lukas, Ovrut, and Waldram [4] within a similar context). Let S^1 be parameterized by $u \in [-l, l]$ with the endpoints identified. The Z_2 symmetry takes $u \rightarrow -u$ and leaves two three dimensional planes fixed. These orbifold fixed planes then are at $u = 0$ and at $u = l$. In this paper we will refer to these as the 3-branes. As in the Hořava-Witten model [1], one of these 3-branes is our observable universe (which we choose without any loss of generality to be at $u = 0$) and the other is a hidden sector that communicates only gravitationally with our visible sector. Thus we define $a(t(\tau)) = e^{\alpha(\tau,u=0)}$ as the scale factor describing the expansion of the 3-brane that is our observable universe where $t(\tau) \equiv \int d\tau \exp(\nu(\tau, u=0))$ is the proper time of a comoving observer (with the brane's position fixed at $u=0$; but see below). The four dimensional Newton's constant is inversely proportional to the distance between these 3-branes,

$$G_N \sim \frac{1}{16\pi m_{pl,5}^3 R(\tau)} \quad (3)$$

where

$$R(\tau) \equiv \int_0^l \exp(\beta(\tau, u)) du. \quad (4)$$

Here $e^{\beta(\tau,u)}$ is the scale factor describing the expansion of the orbifold.

Note that we can always reparameterize the theory such that the 3-brane positions are fixed in orbifold coordinate space. However, in that case, we can no longer have the full freedom of gauge in the (τ, u) plane. For example, we can no longer make a coordinate transformation such that $\nu(\tau, u) = \beta(\tau, u)$ (conformally flat in the τ - u subspace), because such a transformation generally will give the 3-branes a time dependence for their position along the orbifold. However, since we can always reparameterize u by itself or τ by itself, we can rewrite any metric having the property $\beta - \nu = j(u)$ or $\beta - \nu = v(\tau)$ (where $j(u)$ and $v(\tau)$ are real functions) in a conformally flat form (in the τ - u subspace) even with 3-brane coordinate positions fixed.

By varying the action Eq. (1) we find that the bulk equations are

$$\begin{aligned}
e^{-2\nu}T_{00} &= 3e^{-2\beta}(\alpha'\beta' - \alpha'' - 2\alpha'^2) + 3e^{-2\nu}(\dot{\alpha}^2 + \dot{\alpha}\dot{\beta}) \\
e^{-2\alpha}T_{11} &= e^{-2\beta}(3\alpha'^2 - 2\alpha'\beta' + 2\alpha'\nu' - \beta'\nu' + \nu'^2 + 2\alpha'' + \nu'') + \\
&\quad e^{-2\nu}(-3\dot{\alpha}^2 - 2\dot{\alpha}\dot{\beta} - \dot{\beta}^2 + 2\dot{\alpha}\dot{\nu} + \dot{\beta}\dot{\nu} - 2\ddot{\alpha} - \ddot{\beta}) \\
e^{-2\beta}T_{44} &= -3(\alpha'^2 + \alpha'\nu')e^{-2\beta} + 3e^{-2\nu}(2\dot{\alpha}^2 - \dot{\alpha}\dot{\nu} + \ddot{\alpha}) \\
T_{40} &= 3\nu'\dot{\alpha} + 3\alpha'(-\dot{\alpha} + \dot{\beta}) - 3\dot{\alpha}'
\end{aligned} \tag{5}$$

$$T_{40} = 3\nu'\dot{\alpha} + 3\alpha'(-\dot{\alpha} + \dot{\beta}) - 3\dot{\alpha}' \tag{6}$$

where the dots denote partial derivatives with respect to τ while the primes denote partial derivatives with respect to u .

An alternative and equivalent way to think about the five-dimensional space is a boundary picture. Here the coordinate u is restricted to one half of the circle, $u \in [0, l]$ and five dimensional spacetime is written as $[0, l] \times \mathcal{M}_4$. The orbifold fixed planes then turn into boundaries of this five-dimensional space.

We will take the 3-branes to be at fixed orbifold coordinate u . However, the general results of our paper, including the result that ordinary FRW cosmology is modified by the existence of the bulk, would qualitatively hold even if one were to allow the branes to “bend”, i.e., give the location of the 3-branes some u -dependence.

B. Boundary Conditions

The bulk equations only apply in an open region that does not include the boundary. By continuity, the metric on the boundary (brane) is determined by the metric in the bulk. It is the Israel conditions which connect the boundary and the bulk. Hence, the bulk stress energy cannot be set to zero without imposing strong constraints on the energy density of the boundary.

General remarks on Israel Conditions: Here we briefly remind the reader of the Israel conditions (see [3] and references therein), which are relevant when there are boundaries (or equivalently discontinuities of the derivatives of the metric). Alternatively, one can derive equivalent conditions by just inserting a δ function for the matter action (see below) as is done in [4,5].

The Gibbons-Hawking term in Eq. (1) above takes the form

$$S_{GH} = \frac{1}{\kappa_5^2} \sum_{\text{faces}} \int_{\text{face}} d^4x \sqrt{h} K \tag{7}$$

where $K \equiv h^{MN} K_{MN}$ is the trace of the extrinsic curvature $K_{MN} \equiv h_M^P \nabla_P n_N$, $h_{MN} \equiv g_{MN} + n_M n_N$ is the induced metric on the boundary surface, and the vector n^N , orthonormal to the surface, points toward the region which the surface bounds.³

Upon varying S_5 with respect to the metric we find, in addition to the usual

$$\mathcal{R}_{MN} - 1/2 g_{MN} \mathcal{R} = T_{MN} \tag{8}$$

(where $T_{MN} = \frac{2\kappa_5^2}{\sqrt{g}} \frac{\delta S_{\text{erb}}}{\delta g^{MN}}$) within the bulk (which was used in the last subsection), the boundary variation

$$\int \frac{\delta S_{\text{boundary}}}{\delta g^{MN}} \delta g^{MN} + \int \frac{\delta S_{GH}}{\delta g^{MN}} \delta g^{MN} + \frac{-1}{2\kappa_5^2} \int d^5x \sqrt{g} \delta \mathcal{R}_{MN} g^{MN} = 0 \tag{9}$$

where the last integrand is a total divergence and becomes a surface term. This leads to the Israel conditions

$$\sum_{\pm \text{faces}} (K_{MN} - K h_{MN}) = t_{MN}, \tag{10}$$

where the sum over faces is for each side of the boundary surface and we have defined

³The index M , for example, takes integer values from 0 to 4. For example, if we work in the coordinate frame of eqn. (2) with the boundary at a fixed coordinate value, then the fact that $n_m n_N g^{MN} = 1$ implies that $n_4 = \sqrt{-g_{44}} = e^{-\beta}$. Thus $h_{44} = g_{44} + n_4^2 = 0$, while h_{MN} can be nonzero for $M, N < 4$, so that h_{MN} indeed projects onto the boundary. It is then easy to see that only the tangential derivatives ($P = 1, 2, 3$) survive in obtaining K_{MN} .

$$t_{MN} = \frac{2\kappa_5^2}{\sqrt{h}} \frac{\delta S_{boundary}}{\delta h^{MN}} \quad (11)$$

as the energy momentum tensor on the boundary. We will assume that this energy momentum tensor on the boundary can be written in a perfect fluid form. For example,

$$t_0^0 = \kappa_5^2 \rho \quad (12)$$

and

$$t_1^1 = -\kappa_5^2 P, \quad (13)$$

where ρ and P are the energy density and pressure, respectively, measured by a comoving observer. One could write the 00 component of eqn. (10) in the following way, if one also uses eqn. (12). Defining $H_{in} = \frac{1}{a} \frac{da}{dv}$ where $dv = e^\beta du$ (by analogy with the Hubble constant), we can write the 00 Israel condition as $\rho = \frac{3}{\kappa_5^2} \sum_{\pm \text{faces}} H_{in}$.

The Israel condition in eqn. (10) is very similar to $\Delta E_{perp} = 4\pi\sigma$ in electrostatics, where σ is the charge/area on a charged plate and E_{perp} is the component of the electric field perpendicular to the plate. Eqn. (10) intimately relates the energy-momentum on our brane to its extrinsic curvature, i.e., to the way that it is imbedded in the bulk, just as the charge on a plate is intimately connected to the electric field on either side.

Note that there is an alternative way of looking at the problem which gives the same results. Instead of partitioning the orbifold into regions with boundaries (an approach adopted from [3]), one could elevate the integrand of the surface action to a density in five dimensions by inserting a δ function. In that case, we would obtain

$$T_{MN}^{\text{total}} = T_{MN} + t_{MN} \delta(\Sigma) \quad (14)$$

where Σ represents the boundary surface with the appropriate measure for the δ function built in. In lieu of eqns. (8) and (10) we would then use $\mathcal{R}_{MN} - \frac{1}{2}g_{MN}\mathcal{R} = T_{MN}^{\text{total}}$ everywhere.

Note also that the Israel conditions Eq. (10) probe the global structure of the spacetime even though each of the conditions is local. They probe the global structure because they result from the boundary terms arising from integration over all space. Hence, loosely speaking, the topological constraints of the S^1/Z_2 are encoded into local boundary conditions by the Israel conditions.

Note that no additional information, beyond what is contained in Einstein's equations and the Israel conditions, is obtained from considering the Bianchi identities. In the bulk, the Bianchi identities automatically give $\nabla_M T^{MN} = 0$.⁴ On the boundary, $\nabla_M t^{MN} = \sum_{\pm \text{faces}} h^{NM} T_{MPN} n^P$ allows the energy to flow between the boundary and the bulk. Since these equations follow from the Einstein equations and from Eq. (10), there are no additional constraints coming from these equations.

Israel conditions for our model: We can now find the Israel conditions specific to the metric in Eq. (2). Our boundary conditions are found by considering the two 3-branes at $u = 0$ and $u = l$. These are the planes at which we have discontinuities in the normal derivatives of the metric, and hence where the Israel conditions for the boundaries give nontrivial constraints.

As mentioned above, we can always reparameterize the theory such that the 3-branes are fixed in coordinate space. Because we are considering a brane at a Z_2 symmetry fixed plane, from the sum over faces in eqn. (10) we get two identical terms, merely resulting in a factor of two. In a more general spacetime, the Israel conditions in eqn. (10) would still hold, but one would have to explicitly add the contributions from the two sides of the boundary surfaces since these contributions would no longer be equal. With the Z_2 symmetry, the Israel conditions are

$$\mp 6\alpha'|_{\pm} = e^{\beta-2\nu} t_{00} \quad (15)$$

$$\pm \nu'|_{\pm} = \frac{1}{2} e^{\beta-2\alpha} t_{ii} + \frac{1}{3} e^{\beta-2\nu} t_{00} \quad (16)$$

where the $|$ implies that the equation is to be evaluated at one of the two sides of the boundary surface. In each equation, the upper signs are to be used for the face at the boundary plane at $u = 0^+$ while the lower signs correspond

⁴If the bulk contains only one field (say a scalar field), then its equation of motion will be determined by the Bianchi identities; if there are more fields, then other equations of motion are needed as well.

to the face at the boundary plane at $u = l^-$.⁵ Note that because of the Z_2 symmetry defining the orbifold fixed planes, the lower sign also applies to the surface $u = 0^-$, for example. We have also denoted the boundary parallel spatial coordinates \mathbf{x} with the indices i . Using eqn. (12) and $g_{00} = e^{2\nu}$, we see that eqn. (15) can be written $\rho = 6\alpha' m_{pl,5}^3 e^{-\beta}$.

As mentioned above, if one makes a particular gauge choice such as $\nu(\tau, u) = \beta(\tau, u)$, then one must allow the possibility of a moving 3-brane. Only if $\beta - \nu$ is independent of time or independent of u can one choose $\nu(\tau, u) = \beta(\tau, u)$ with fixed brane positions without any loss of generality. The Israel conditions for a 3-brane having a time dependent coordinate $r(\tau)$ are

$$t_{00} = \mp 6e^{2(\nu-\beta)} \left[\frac{(e^{2\nu}\alpha' + e^{2\beta}\dot{\alpha}\dot{r})}{(e^{2\nu} - e^{2\beta}\dot{r}^2)\sqrt{e^{-2\beta} - e^{-2\nu}\dot{r}^2}} \right] \quad (17)$$

$$t_{11} = \pm e^{2\alpha} \left[\frac{\ddot{r} + 2e^{2(\nu-\beta)}\alpha' + \dot{r}^2(\beta' - 2\alpha') + e^{2(\nu-\beta)}\nu' - 2\dot{r}^2\nu' + 2\dot{r}(\dot{\alpha} + \dot{\beta}) - e^{2(\beta-\nu)}\dot{r}^3(\dot{\beta} + 2\dot{\alpha}) - \dot{r}\dot{\nu}}{(e^{2\nu} - e^{2\beta}\dot{r}^2)\sqrt{e^{-2\beta} - e^{-2\nu}\dot{r}^2}} \right]. \quad (18)$$

As explained before, with a proper choice of coordinates, the 3-brane positions can be fixed ($\dot{r} = 0$) at the coordinates that we have chosen for all bulk coordinate time. However, if we insist instead that $\nu = \beta$, then this apparent loss of a functional degree of freedom will reappear in the nonzero \dot{r} except under certain conditions previously discussed.

III. NONSTANDARD EVOLUTION OF OUR 3-BRANE

From the 4-dimensional point of view of a low energy observer living on the 3-brane, the effect of the extra dimensions can be considered as the coupling of some extra moduli fields to the boundary energy density. The distinguishing feature of the brane scenario is that these extra fields arise from the gravitational coupling to the extra dimensions and hence must obey the boundary equations Eq. (15) and Eq. (16). From Eq. (5), not only is the usual four-dimensional Friedmann equation relating the energy density and the Hubble speed modified, but Eq. (16) restricts the possible relationship even further. In general, there seem to be no restrictions of the stress energy tensor that drive the energy density confined on the 3-brane to behave as H^2 (an explicit illustrative example of this will be given later in the next section V). Hence, the relationship between the energy density of matter confined to the brane and the expansion scale factor will generally be different from the standard cosmology.

Let us consider why one does not generically recover the standard evolution for our observable universe: i.e. why in general H^2 will not be proportional to the energy density confined to the observable sector 3-brane. The main point is that the behavior of H and thus the time evolution of our observable universe can be controlled by the moduli from the extra dimensions and the attending boundary conditions of the spacetime, rather than by what is on our brane. One can think of this as a version of the moduli problem, which states that our 3-brane's stress energy is subdominant compared to the effect of the moduli coming from extra dimensions on our universe's history.

The above statement is most transparent when one considers the full five dimensional Einstein's equations rather than the dimensionally reduced 4-dimensional effective action. For example, consider the case where there is no stress-energy in the bulk, i.e. $T_{\mu\nu} = 0$. Still, the behavior of the extra dimension is determined by the 3-brane stress energy $t_{\mu\nu}$, and the derivatives along the extra dimensions of metric components act as energy sources for the Friedmann equation governing the expansion of the observable 3-brane. Consider, for example, Eq. (5). From a higher dimensional point of view, the first of the equations in Eq. (5) clearly shows the contributions to H that are independent of T_{00} , the bulk energy density. For example, we can take β to be a constant ($\dot{\beta} = \beta' = 0$) and set $T_{00} = \alpha'' = 0$. Then, using the Israel condition in Eq. (15), as well as Eq. (3), Eq. (12), and the fact that $H = \frac{da/dt}{a} = e^{-\nu}\dot{\alpha}$, we obtain the result

$$H = \frac{8\pi G_N}{3}(\sqrt{2}l\rho), \quad (19)$$

which shows a linear relationship between ρ and H . Hence, even if the bulk is empty, we generally will not recover the standard FRW cosmology. This unusual relationship has also been discussed by Ref. [5], which considered the 4-4 component of the Einstein's equation with the assumption

⁵Here, for example, $u = 0^+$ corresponds to approaching $u = 0$ from the positive side ($u > 0$).

$$\rho_{bulk} \ll \frac{\rho_{brane}^2}{m_{pl,5}^3}, \quad (20)$$

where ρ_{brane} is the energy density on the brane and ρ_{bulk} is the energy density on the bulk, and showed that an unusual form of the Friedmann equation ensues (a linear relationship between H and ρ).

In general, almost any relationship between ρ and H is possible (e.g. such as $\rho \sim H^q$). One can see this, for example, by using the fact that α'' and β' on the 3-brane are not well constrained. From the four dimensional effective theory point of view, this is the same as saying that the moduli not confined to our brane can contribute to the energy density such that nearly arbitrary time dependence for the expansion scale factor a may be achieved. More explicitly, suppose we seek a solution of the form

$$\rho = \mu H^q \quad (21)$$

where μ is a constant which we can parameterize as $M^{4-q}c$ where M is a mass scale and c is a dimensionless constant. In that case, the Israel equations, at $u = 0$ for example, become

$$-\nu'(\tau, u=0) = 3\alpha'(\tau, u=0)(w(\tau) + \frac{2}{3}) \quad (22)$$

$$6\alpha'(\tau, u=0)e^{-\beta(\tau, u=0)} = \mu\kappa_5^2(e^{-\nu(\tau, u=0)}\dot{\alpha}(\tau, u=0))^q \quad (23)$$

where $w = -t_i^i/t_0^0$ is the equation of state. Note that because these constraint equations are only evaluated at a point on the orbifold, one can consider $\nu(\tau, u=0)$ and $\nu'(\tau, u=0)$ as independent functions. Similarly, $\alpha(\tau, u=0)$ and $\alpha'(\tau, u=0)$ can be considered independent functions.

For example, we can construct a solution for any value of q with $w \equiv P/\rho = 0$ (pressureless fluid) and $\rho \propto 1/a^3(\tau, u=0) = \exp(-3\alpha(\tau, u=0))$ as follows:

$$\beta(\tau, u) = \nu(\tau, u) = \frac{\kappa_5^2}{3} \left[\frac{q}{3\tau} \right]^q \mu u \quad (24)$$

$$\alpha(\tau, u) = \frac{-\kappa_5^2\mu}{6} \left[\frac{q}{3\tau} \right]^q (F(u) - F(u=0)) + \ln(\lambda) + \frac{q}{3}\ln\left(\frac{\tau}{\tau_0}\right) \quad (25)$$

where $F(u)$ is any smooth function satisfying $F'(u=0) = 1$ and λ , τ_0 , and μ are constants. For example, suppose $q = 2$ (corresponding to the usual FRW relationship) and $F(u) = u$. The resulting stress energy tensor components are

$$T_0^0 = \frac{-4 \exp(-\frac{8\kappa_5^2\mu u}{27\tau^2})(-81\tau^4 + 4\kappa_5^4\mu^2(\tau^2 + u^2))}{243\tau^6} \quad (26)$$

$$T_1^1 = \frac{-4 \exp(-\frac{8\kappa_5^2\mu u}{27\tau^2})\kappa_5^2\mu(-36\tau^2u + \kappa_5^2\mu(\tau^2 - 4u^2))}{243\tau^6} \quad (27)$$

$$T_4^4 = \frac{-2 \exp(-\frac{8\kappa_5^2\mu u}{27\tau^2})(81\tau^4 + 54\kappa_5^2\mu\tau^2u + 2\kappa_5^4\mu^2(\tau^2 + 16u^2))}{243\tau^6} \quad (28)$$

$$T_{40} = \frac{40\kappa_5^4\mu^2u}{243\tau^5} \quad (29)$$

$$\rho = \frac{4\mu}{9\tau^2} \quad (30)$$

$$P = 0 \quad (31)$$

Hence, with a nontrivial time dependent bulk stress energy tensor in the bulk, one can recover the FRW expansion on the boundary. Note that in this particular example, there is no energy flowing between the brane and the bulk.

Of course, once a specific model is written down with specific choices for the stress-energy tensor components $T_{\mu\nu}$, we will be able to solve Einstein's equations together with the boundary conditions to calculate exactly which q results in $\rho \sim H^q$. At present we can only point out that different choices of $T_{\mu\nu}$ will give rise to different values of q , and generically will produce modifications to the standard FRW results.

Let us now consider the modifications to the standard FRW cosmology from a purely four dimensional effective action point of view. The total action, Eq. (1), can be dimensionally reduced to

$$S_4 \approx \frac{-R}{2\kappa_5^2} \int d^4x \sqrt{g_4}(\mathcal{R}_4 + \Delta\mathcal{R}) + S_{orb4} + S_{boundary1} + S_{boundary2} \quad (32)$$

where $\Delta\mathcal{R}$ is the gravitational moduli contribution that arises from dimensionally reducing the 5-dimensional Ricci scalar, S_{orb4} is the contribution coming from the bulk field, and the remaining action terms correspond to the brane-confined field contributions. In the Hořava-Witten model, $S_{boundary1}$ corresponds to the contribution from the observable sector and $S_{boundary2}$ corresponds to the contribution from the hidden sector. What we are calling the moduli problem is that $S_{boundary2}$, S_{orb4} , and $\Delta\mathcal{R}$ contributions can dominate the energy density determining the expansion rate of the universe. For a particular orbifold geometry about which the dimensional reduction is taken, the stress energy contributions coming from $\Delta\mathcal{R}$, $S_{boundary1}$, $S_{boundary2}$ and S_{orb4} are related through the Israel conditions and the Einstein equations. In Ref. [5], a linear relationship between H and ρ arising from this moduli dominance was emphasized, particularly when $S_{orb4} = 0$. We would like to merely point out that in general, any relationship between H and ρ may result if all the moduli contributions (i.e. $\Delta\mathcal{R}$, $S_{boundary2}$, and S_{orb4}) are taken into account.

IV. CONSTRAINTS

In the last section, we saw that the ordinary matter confined to our brane does not (alone) determine the expansion rate of our universe H ; in contrast the dilution of the energy density and the pressure of the ordinary matter proceeds as usual as the universe expands. Hence, any physical observable that measures the expansion rate independently of the “ordinary” energy density will be able to test this 3-brane scenario. To illustrate this constraint, we will consider big bang nucleosynthesis (BBN), structure formation, and the age of the universe.

Big Bang Nucleosynthesis: As discussed above, generically one obtains modifications to the cosmological standard model relationship between evolution of the scale factor and energy density on our brane. These modifications will drastically alter nucleosynthesis. In BBN, a given reaction rate depends directly on the energy density (the photon number density, for example) while the freeze out is governed by the ratio of the Hubble speed to the reaction rate. Hence, the energy density time variation can be compared with the universe expansion rate since the ratios of the various light element abundances measure the various temperatures at which different elements freeze out.

At the high temperatures in the early universe, the ratio of neutrons to protons is determined by its thermal equilibrium value,

$$n/p = e^{-Q/T}, \quad T \geq T_F, \quad (33)$$

where the neutron-proton mass difference $Q = 1.293$ MeV. Neutrons drop out of equilibrium below a freeze-out temperature T_F , where the weak interaction rates can no longer keep up with the expansion of the universe. Below T_F the n/p ratio continues to fall due to β -decay on the time scale of the neutron half-life τ_n . In the standard BBN scenario, nucleosynthesis begins at a temperature approximately given by

$$T_D = 2.2 \frac{\text{MeV}}{-\ln\eta} \quad (34)$$

where η is the baryon to photon ratio. Once T_D is reached, deuterium becomes stable against photodissociation and nucleosynthesis takes place very rapidly, efficiently converting essentially all of the available neutrons into He^4 . In this approximation, the primordial helium abundance Y_p is given by

$$Y_p = \left(\frac{2n}{n+p}\right)_D = \left(\frac{2n}{n+p}\right)_F \exp[-\Gamma(t_D - t_F)] \sim \frac{2e^{-\Gamma t_D}}{1 + \exp[Q/T_F]} \quad (35)$$

where the final approximation is valid since $\Gamma^{-1} = \tau_n/\ln 2 \gg t_F \sim 1\text{sec}$.

Now we consider the modification of these results when the equation $H^2 \sim \rho/m_{pl}^2$ is modified. As an example, let us consider the following modification to standard 4-dimensional Einstein equations:

$$H = \rho/M^3, \quad (36)$$

where M is some mass scale, perhaps generically the Planck mass. Such a relationship was discussed above and is also found in the toy model discussed in the next section. It was also found by Lukas, Ovrut, and Waldram [4] in their nonlinear case as well as by [5] in a similar context. If we assume that the scale factor expansion is a power law, we then have $H \sim 1/t$. In a radiation dominated universe, $\rho \sim T^4$. With Eq. (36), one can then find that the scale factor grows as $a \sim t^{1/4}$ and the modified temperature-time relation is $T \sim t^{-1/4} M^{3/4}$. Recall that the freeze-out of neutrons and protons occurs when a typical $n \leftrightarrow p$ weak interaction rate $\Gamma \sim G_F^2 T_F^5$ is equal to the expansion rate H . Here $H \sim T^4/M^3$ (as opposed to the usual T^2/m_{pl} of the standard BBN model). Thus we find

$$T_F = \bar{T}_F^3(m_{pl}/M^3), \quad (37)$$

where an overbar indicates the standard BBN model value. For example, if $M = m_{pl} = 10^{19}\text{GeV}$, we have $T_F = 10^{-44}\bar{T}_F$. This model has the novel feature that $T_D > T_F$, so that the exponential factor in the numerator of Eq. (35) is not there. However, both temperatures are significantly lower than the usual \bar{T}_F , so that the production of He^4 is driven to zero!

In addition to the specific case considered above of $H \propto \rho$, one can allow almost any q in the equation $\rho \propto H^q$, depending on the $T_{\mu\nu}$ in the bulk. To analyze a variety of possibilities, we consider power law growth of the scale factor both in the matter and the radiation dominated regimes, and allow the power law to be a free parameter. Thus we take the scale factor on the 3-brane to be $e^{\alpha(\tau, u=0)} \sim (t(\tau, u=0)/t_i)^n$ for the radiation dominated era, where t_i is a constant and $t(\tau, u=0)$ is the proper time of the comoving observer on the 3-brane. During matter domination we take the power law index to be m instead of n (with m and n usually different). Consider the neutron-proton interconversion interactions freezing out. By setting the reaction rate equal to the Hubble expansion rate, to within an order of magnitude, the freeze out temperature then is found to be

$$T_F \sim \left(G_F^2 \frac{T_0^{1/m}}{H_0} (\rho_{0m}/\rho_{0r})^{[1/n-1/m]} \right)^{\frac{1}{(1/m-5)}} \quad (38)$$

where T_0 is the temperature of the cosmic background photons today H_0 is the Hubble speed today, ρ_{0m} is the matter density today, and ρ_{0r} is the radiation density today.

Compare for example the two possible cases: 1. $\{n = 1/2, m = 2/3\}$ (usual FRW scenario) and 2. $\{n = 1/4, m = 1/3\}$ (a possible brane scenario). For case 1, we obtain a crude estimate $T_F \sim 1 \text{ MeV}$ while for case 2, we obtain a crude estimate of $T_F \sim 10^{14} \text{ GeV}$. Hence, even if this crude estimate were orders of magnitude off,⁶ we will obtain a large measurable discrepancy for the resulting element abundances since the neutron to proton ratio is constrained to a precision of less than 1%.

Structure Formation: Consider the Jeans analysis of a fluid overdensity within a universe with $a \sim t^m$ during the matter dominated epoch. To obtain an idea of the type of effects that we may find, instead of analyzing the exact perturbation equations with metric perturbations included,⁷ we will merely modify the time dependence of the scale factor in the usual Jeans analysis equation valid for standard FRW cosmology. We assume the usual long wavelength limit and consider only one component nonrelativistic matter fluid. Normalizing the background energy density and the Hubble speed to approximately what we measure today (i.e. $\rho_0 \approx 3/(8\pi)M_{pl}^2 H_0^2$), the fluid overdensity equation can be written as

$$\delta''(x) + \frac{2m}{x}\delta'(x) - \frac{3m^2}{2}x^{-3m}\delta = 0 \quad (39)$$

where $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$ is the fluid overdensity and $x \equiv t/t_0$ with t_0 denoting the time today. This equation can generally be solved in terms of Bessel functions for constant m . In the standard FRW cosmology with matter domination, $m = 2/3$, and there is one solution to δ which grows as the scale factor $\delta \sim a \sim t^{2/3}$ and the other decreasing as $1/x$. However, take for example the case where $\rho \sim H$ which is equivalent to setting $m = 1/3$ if $\rho \propto a^{-3}$. In this case, there will be two growing solutions for δ , one having the form $1 + x/4 + \dots$ and the other $x^{1/3}(1 + x/8 + \dots)$. This agrees with the expectation that in a universe that is expanding more slowly, the overdensity will grow more quickly with the scale factor.

Extra growth of the perturbations compared to the standard picture might enable some new models to be considered, that are ruled out in the standard model. For example, one might be able to better tolerate a smooth component in the universe (which ordinarily suppresses perturbation growth) and still get enough growth of the fluctuations to get structure today. For example, particles that decay to radiation might be able to lead to a longer radiation dominated universe subsequent to the initial matter/radiation equality. However, the standard picture of cold dark matter might well be in trouble with altered growth of the scale factor. With this altered growth, calculation of the integrated Sachs Wolfe effect and normalization to the COBE data would undoubtedly lead to anisotropies in the power spectrum which are less scale invariant than the standard model, such that one would not be able to match the data on all different scales.

⁶The actual freeze out temperature will be lower, making the proton-neutron relative abundance ratio (which is approximately $\exp(-1\text{MeV}/T_F)$) discrepancy from the observed smaller.

⁷This more careful treatment is in progress.

The time dependence of the scale factor for a given type of ordinary energy density ρ can be tested by considering the location of where the collapse and diffusion processed power spectrum starts to flatten out. This flattening occurs in the standard CDM scenarios because a particular nonrelativistic matter fluctuation Fourier mode grows only after the matter domination phase begins and the mode wavelength enters the horizon. The horizon length at the time of matter radiation equality is approximately t_{eq} . For wavelengths longer than t_{eq} (at that time), the modes are outside of the horizon and will not grow until they enter the horizon. Hence, today, the mode's amplitude will be suppressed with respect to the mode that grew starting at time t_{eq} . Modes that had already entered the horizon by the time t_{eq} will grow together, and hence the spectrum will be flat in that region.

With a different time dependence for the scale factor $a \sim t^m$, the location of this bend in the power spectrum will be shifted from λ_1 to λ_2

$$\lambda_2 = \frac{3m}{2} \left(\frac{\rho_{0R}}{\rho_{0M}} \right)^{\frac{1}{m} - \frac{3}{2}} \lambda_1. \quad (40)$$

For the case $\rho \propto H$ ($m=1/3$), we have $\lambda_2 = 10^{-7} \lambda_1$ if we assume that ρ_{0M} is order of the critical density (*i.e.*, $8.1 \times 10^{-47} \text{GeV}^4$).

Age of the Universe With the time dependence of the scale factor altered, the age of the universe may be modified. For example, if we assume that structures form during a period after the matter-radiation equality with $a \sim t^m$ and that the scale factor at matter-radiation equality a_{eq} is much smaller than today's scale factor a_0 , *i.e.* $(a_0/a_{eq}) \gg 1$, then the time period of structure formation is given by $t = m/H$ where H is the Hubble speed today. Hence the lower bound on the age of the oldest stars in a globular clusters may be used to constrain m . Taking the value of [9] of $t > 10.2$ Gyr, we find

$$m > h \quad (41)$$

where the Hubble speed is $H = 100h \text{ Km/s/Mpc}$. This would rule out $\rho \propto H$, since in that case $m = 1/3$, whereas observations of the Hubble constant find that $h > 0.5$.

V. DIFFICULTIES OF STATIONARY PLANCK MASS COSMOLOGY

We have discussed the altered relationship between the energy density and the Hubble speed in the 3-brane scenario compared to the standard cosmology. Another difference from the standard cosmology is that Newton's gravitational coupling constant can have a time dependence in the Hořava-Witten scenario because it is inversely proportional to the orbifold proper length. Naively, one would, in constructing a cosmological scenario, then try to freeze the orbifold radius. However, as we will see, this turns out to be unphysical, at least in the case where the metric can be written in a conformally flat form in the $\tau - u$ subspace. If one allows the orbifold length to vary slowly but within the limits of astrophysical bounds, we still expect some constraint although it is not as severe. We will explicitly illustrate this with a toy model. This difficulty of freezing the four dimensional Planck's constant is yet another manifestation of the cosmological constant problem.

We will consider the stress-energy tensor in the comoving frame. As in Eq. (12) and Eq. (13), we denote the energy density and the pressure as $t_0^0 = \kappa_5^2 \rho$ and $t_i^i = -\kappa_5^2 P$ respectively. Hence, Eq. (16) can be rewritten as

$$\nu'|_{\pm} = \pm \left(\frac{1}{2}w + \frac{1}{3} \right) t_0^0 e^{\beta} \quad (42)$$

where we have introduced a time dependent function $w \equiv P/\rho$ defining the equation of state for the stress energy. Now, suppose we have a metric with the property that $\nu - \beta$ is purely a function of u or τ ; any such metric can be put into the conformally flat form in the $\tau - u$ subspace. In other words, for any such metric we can effectively take $\nu = \beta$ in Eq.(2). Given such a metric, we will now attempt to fix the orbifold radius and consequently the four-dimensional Planck mass by taking β to be time independent. Since Eq. (42) holds for all time, we can impose the condition

$$\frac{d}{dt} \beta'|_{\pm} = 0 \quad (43)$$

where, since β is time independent, $t = e^{\beta(u)} \tau$. Solving Eq. (43) and Eq. (42) for $w(t)$, we find

$$w(t) = \frac{-2}{3} + \frac{w(t_i) + 2/3}{(\rho(t)/\rho(t_i))}, \quad (44)$$

where t_i represents some initial time.

Hence, for all energy densities that decrease with time and $w(t_i) \neq -2/3$, our equation of state grows in magnitude without bound. For the usual macroscopic stress energy consisting of homogeneous scalar fields and fermionic particles bound to the brane, the equation of state is constrained to $-1 < w(t) < 1$. Hence, Eq. (44) most likely will violate any realistic equation of state, and the four dimensional Planck mass cannot be fixed to a constant value (e.g. with a potential). We comment briefly on the possibility that $w(t) = w(t_i) = -2/3$ exactly, the only value that allows the radius to stabilize for decreasing energy density. We note that this value has been proposed (Wang *et al* [10]) as the best fit value to the combination of a number of data sets, including recent supernovae observations as well as various measures of large scale structure and the microwave background. However, in the context of Eq. (44), obtaining this value would require either some unknown dynamical mechanism to drive the number to $w = -2/3$, or an uncomfortable fine tuning. We can see that, for a universe with a decreasing energy density, the denominator in the second term continues to decrease with time and drives the second term away from the phenomenologically acceptable value $-1 < w < 1$, unless $w(t_i)$ in the numerator is fine tuned to be arbitrarily close to $-2/3$.

We wish to comment that the difficulty we find in attempting to stabilize the radius of the extra dimensions to a fixed value is similar to the problem of trying to have a static Einstein universe. Both require fine tuning. While a static universe can be achieved with a proper parameter choice, it is unstable to perturbations about these values; similarly, a stabilized radius of the extra dimension is unstable to slight deviation of $w(t_i)$ from the precise value of $2/3$. Although we have only been able to prove the fine tuning required for radius stabilization for specific metrics in the Hořava Witten scenario, we suspect that such a result may generalize to any attempt to stabilize the radius in the context of an extra dimension in which gravitons propagate⁸.

A way to avoid these conclusions and still to satisfy Eq. (44) is if $\rho(t)$ on the observable 3-brane is dominated by a constant vacuum energy (an effective cosmological constant) for all time. Then $\rho(t) = \rho(t_i)$, $w(t) = w(t_i) = -1$, and Eq. (44) is trivially satisfied. Such an energy density on the brane does not automatically lead to exponential expansion of the brane if there is a nonlocal contribution canceling the effect of the 3-brane cosmological constant, and hence this case can possibly be made to match today's universe. However, generically Eq. (44) disagrees with the evolution of our observable universe.

The condition that the metric on the orbifold be absolutely static is clearly too stringent. The radius of the orbifold must be allowed to change. Then the only experimental constraint on the variation of the orbifold length comes from the bound on the variation of the Newton's constant (see for example [12] and references therein). The most conservative bound is approximately

$$\frac{dG_N}{dt}/G_N < 6 \times 10^{-11} \text{yr}^{-1}, \quad (45)$$

which comes from the spin down rate of pulsars. Although this bound is only measured on astrophysical time scales, we will assume it to be valid for cosmological time scales. We remind the reader that G_N is inversely proportional to the distance between 3-branes (Eq. (3)), which is obtained via Eq. (4) as an integral over the entire bulk. Hence the best way to avoid the constraints of Eq. (43) and Eq. (44), while still satisfying the observations on dG_N/dt , is to have a large time variation of the orbifold metric near the brane and vanishing time variation elsewhere throughout the bulk. We will construct such an example shortly.

The most general constraint from Eq. (16) arises as a result of w being of order unity. This implies that, at $u = 0$,

$$|e^{-\beta} \beta' l| \sim \epsilon \equiv |t_0^0 l| \sim 10^{-114} \quad (46)$$

where $l \sim 10^{-15} \text{GeV}^{-1}$ is the physical length scale of the orbifold appropriate for the Hořava-Witten model considered in [4]. The numerical estimate in Eq. (46) is made by taking the energy density to be $\rho \sim 10^{-29} \text{ gm cm}^{-3}$ as can be roughly inferred from rotation curves of galaxies (we motivate this number with rotation curves rather than with the typical critical density of standard cosmology because the latter may change in these models). This condition says that, very near the brane, the orbifold scale factor remains approximately independent of the transverse distance off the brane. Below we will allow the physical length scale of the orbifold l to be as large as a mm and find that our conclusions are relatively unchanged.

Now we will construct an explicit example which demonstrates that the time variation bound and Eq. (46) can be simultaneously satisfied by allowing the orbifold length to have a time variation. However, as we will see, the model is still unnatural. Consider the following ansatz:

⁸Remarks on the difficulty of stabilizing the radius in extra dimensions can also be found in [11], which appeared shortly after this work

$$\beta \equiv \frac{2\tau h}{\lambda} e^{-\lambda l/2} \cosh[\lambda(u - l/2)] \quad (47)$$

where $\lambda l \gg 1$. Note that this ansatz fits the optimal situation described below Eq. (45) in that most of the time variation is near the boundary: dG_N/dt is determined by $d\beta/dt$ with β (and its time derivative) biggest at the boundaries. We define the orbifold proper length to be

$$\int_0^l e^{\beta(\tau(t), u)} du \quad (48)$$

where t is the comoving time on the brane. Then, from equations Eq. (3), Eq. (45), Eq. (47), and Eq. (48), the time variation bound on the Newton's constant translates into

$$\frac{2|h|}{l\lambda^2} < c_1 \quad (49)$$

where $c_1 = 6 \times 10^{-11} \text{yr}^{-1}$. Dropping factors of order unity, combining Eq. (49) and Eq. (46), we find the constraint on h to be

$$\frac{2}{c_1 \epsilon^2 t^2 l} < |h| \quad (50)$$

where $\epsilon = 10^{-114}$ for $l \sim 10^{-15} \text{GeV}^{-1}$. For example, if one takes $t = 10^{10} \text{yr}$, then if $|h| > 10^{233} \text{GeV/yr}$, the time variation constraint on the Newton's constant is satisfied. Note that if one took l as large as 1 mm, the parameter will still be unnatural; since $\epsilon \propto l$, the bound on $h \propto 1/l^3$ and we would require $|h| > 10^{149} \text{GeV/yr}$. Of course, such a large dimensionful constant most likely is unphysical. Incidentally, for $l \sim 10^{-15} \text{GeV}^{-1}$, $\lambda > 10^{127} \text{GeV}$ which is much greater than $1/l$ as was assumed below Eq. (47). Note also that although Eq. (49) seems to imply an upper bound on the value of $|h|$, instead a lower bound arises because λ in Eq. (46) has to adjust as $|h|$ is varied to keep the equation of state reasonable. This requirement of such an unphysically large number for h is a result of $t_0^0 l$ being very small. This in turn is just a restatement of the cosmological constant problem in the sense that there is a severe mismatch between the natural scale $1/(\kappa_5^2 l)$ and the energy density on the brane. One way of accommodating the severe mismatch is to have the long age of the universe (corresponding to the small energy density of the universe) built into the solution itself. Indeed, Eqs. (24) and (25) with $q = 1$ give such an example, where $1/\tau \sim \mathcal{O}(1/10^{10} \text{yr})$ supplies the unnaturally small (compared to $1/(\kappa_5^2 l)^{1/4}$) energy scale.

A cosmological example of our 3-brane for $\nu = \beta$

Now we turn to evolution of the 3-brane when $\nu = \beta$. We consider the other Israel equation Eq. (15) which specifies ρ once α is given. Suppose we have the brane expanding as a power law in comoving time. Thus, we write

$$\alpha = \ln f(u) + m \ln(t(\tau, u)/t_i) \quad (51)$$

where m is the exponent of the power law and t_i is a constant. Using the ansatz for β in Eq. (47), we then find

$$\rho = A(u) \frac{1}{t} \ln(1 + t\theta) + B(u) \frac{1}{1 + t\theta} \quad (52)$$

where $A(u)$ and $B(u)$ are some functions of u and

$$\theta \equiv \beta/\tau. \quad (53)$$

Note that, as $t \rightarrow \infty$, $\rho \rightarrow (A + B/\theta)/t \propto H$. This is in contrast with the FRW cosmology where $\rho \propto 1/t^2 \propto H^2$ independently of the equation of state, as long as the scale factor is a power law with respect to time. Hence, we have a novel feature of the energy density scaling like the Hubble speed instead of its squared. As discussed in the previous section, such a relationship between the energy density and the time causes many cosmological problems. Furthermore, if one writes down the equation of state corresponding to this cosmology, one will see that w approaches a constant value that is determined, e.g., by the energy density at some initial time.

Note that this behavior relies strongly on the assumption $\nu = \beta$, which cannot always be made to be true if the orbifold fixed planes have time independent coordinate positions. Indeed, without the choice $\nu = \beta$, the physical time derivative taken to obtain the Hubble speed has no general relationship with the physical spatial (orbifold directional) derivative taken to obtain the energy density ρ . Note that the bulk equations are largely unconstrained because of the freedom we have in choosing $\alpha(\tau, u)$ for u away from the boundaries. Even with the simple power law expansion of the scale factor discussed in this section, $f(u)$ in Eq. (51) is unconstrained except at $u = 0$.

VI. CAUSALITY AND THE HORIZON PROBLEM

Interesting new avenues for solutions to the horizon problem arise in the context of extra dimensions. Here we confine ourselves to a few remarks and reserve a more complete discussion to a work in progress. Previous work by Freese and Levin [13] addressed the horizon problem in some generality, of course in a four-dimensional context. We discuss here one aspect of the generalization of that work to higher dimensions. The major point we wish to make is that the application of standard inflation requirements, such as the requirement of 60 e-foldings, may be modified in the context of extra dimensions as some of the assumptions that went into obtaining this requirement are relaxed.

Note that the discussion in this section is true for any model in which the length of the extra dimension(s) determines the four-dimensional gravitational coupling. For example, it applies to periodic boundary conditions considered by many authors including [14].

An estimate of the size of the observable universe today is given by the distance light could travel between photon decoupling and now, $d_{obs} \sim a(t_o) \int_{t_{dec}}^{t_o} dt/a(t) = O(1) \times (t_o - t_{dec})$ for $a \propto t^p$ and $p = O(1)$ between t_{dec} and t_o . Here a is the scale factor. We can compare the comoving size of the observable universe to the comoving size of a causally connected region at some earlier time t_c : $d_{hor}(t_c)/a(t_c) = \int_0^{t_c} \frac{dt}{a(t)}$. The observable universe today fits inside a causally connected region at t_c if

$$\frac{d_{hor}(t_c)}{a(t_c)} \geq \frac{d_{obs}}{a(t_o)}. \quad (54)$$

Here, subscript o refers to today and subscript c to some early time. If condition Eq. (54) is met, then the horizon size at t_c (before nucleosynthesis) is large enough to allow for a causal explanation of the smoothness of the universe today. Note that more creative explanations of large scale smoothness may not involve comparing these two patches. For example, in the context of the brane scenarios, one might imagine that two regions of our observable universe which seem to be causally disconnected might in fact have talked to each other because of a geodesic between them that went off our brane, into the bulk, and then back onto our brane at some distant point. Such alternatives are certainly worth investigating further. In the remainder of our discussion here we restrict ourselves to the case where Eq. (54) is relevant; this is certainly the case for all the brane and boundary inflation models considered to date.

For power law expansion of the scale factor both before t_c and after t_{dec} (which may or may not be the case), we can take $t \sim H^{-1}$ during these periods. The causality condition Eq. (54) then becomes

$$\frac{1}{a_c H_c} \geq \frac{1}{H_o a_o}. \quad (55)$$

We take the Hubble constant today to be

$$H_o = \alpha_o^{1/2} T_o^2 / m_{pl}(t_o), \quad (56)$$

where $\alpha_o = (8\pi/3)(\pi^2/30)g_*(t_o)\eta_o$, g_* is the number of relativistic degrees of freedom and $\eta(t_o) \sim 10^4 - 10^5$ is the ratio today of the energy density in matter to that in radiation.

Following Freese and Levin [13], one can further simplify this condition by making a series of simplifying assumptions. First, we assume for now that the entropy on the 3-brane both at time t_c and today obeys $S \propto (aT)^3$ such that we can write a in Eq. (54) in terms of the entropy. A work in progress is reexamining this condition.

A further assumption is always made in obtaining requirements for inflation to succeed. It is that the Hubble speed at the early time t_c is given by

$$H_c \sim T_c^2 / m_{pl}(t_c), \quad (57)$$

where T_c is some mass scale, possibly the temperature, at time t_c . As can be seen by the earlier results in this current paper, this assumption may not always hold. Thus one should in principle always go back to the original equation Eq. (54) to check that any particular inflation proposal really does solve the horizon problem. However, for the models that have been heretofore proposed, the usual requirements used are at least qualitatively similar to the actual requirements implied by above equations.

Even if above assumptions hold, yet another fact must be considered: the changing four dimensional Planck mass. Using Eqs. (55), (56), (57), and the entropy relation, the causality condition becomes

$$\frac{m_{pl}(t_c)}{m_{pl}(t_o)} \left(\frac{S_o}{S_c} \right)^{1/3} \geq 10^{-2} \frac{T_c}{T_o}. \quad (58)$$

One can now invoke the idea of changing the four dimensional Planck mass in lieu of changing the entropy in order to solve the horizon problem, as has been suggested by Levin and Freese [15] and as is studied in the context of string theory by Veneziano [16] and collaborators. Here, in the context of extra dimensions, if the compactified dimension that determines the strength of gravity shrinks with time, one may be able to use the changing four dimensional Planck mass to explain causality as in Eq. (58) (see also Riotto [8]).

In the context of inflation, one must also be careful about the changing four dimensional Planck mass due to the changing size of the extra dimension in obtaining the appropriate causality condition. Suppose inflation ends at time t_e . Assuming that entropy increases by the end of inflation while it remains constant after inflation, we can then rewrite the condition Eq. (58) as

$$\frac{a_e}{a_c} \geq 10^{-2} \frac{T_c^2}{T_e T_o} \left(\frac{m_{pl}(t_o)}{m_{pl}(t_c)} \right). \quad (59)$$

It is this last factor of $m_{pl}(t_o)/m_{pl}(t_c)$ that must not be forgotten, if the value of the four dimensional Planck mass changes between the time of inflation and now. One way to think about this is that in taking our observable universe today and rescaling it to some early time (to see if it fits inside a causal radius), the size of the observable patch may be bigger (or smaller) than one would naively think because of the difference in the Planck masses. For example, in the inflation model of [14] inflation takes place at an early stage while the radius of compactification is very small and the Planck mass much smaller than today. If one takes $m_{pl}(t_c) = \text{TeV}$, for example, then one needs the scale factor to grow by an additional factor of $10^{19} \text{GeV}/\text{TeV} = 10^{16}$ compared to the standard inflationary result with constant Planck mass. Instead of the usual 60 e-foldings one needs 100. This is not a significant difference, since it is usually just as easy to get 100 e-foldings as 60. Still, we caution that there are many assumptions that go into the calculation of requirements for inflationary models, and these should in principle be checked.

VII. CONCLUSIONS

In this work, we have explored the general cosmology of a flat, homogeneous and isotropic 3-brane located at the Z_2 symmetry fixed plane of a Z_2 symmetric five dimensional spacetime as in the Hořava-Witten model compactified on a Calabi-Yau manifold. We have found an important manifestation of the moduli problem: the usual relationship between the FRW scale factor and the energy density on our 3-brane is modified from $\rho \sim H^2$ to almost any relationship between ρ and H , such as $\rho \sim H^q$ with $q \neq 2$. Hence, in these models, one does not generically recover the standard FRW evolution for our observable universe. The time evolution of our observable universe can be controlled by the moduli from the extra dimensions and the attending boundary conditions of the spacetime, rather than by the energy density on our brane. These modifications persist even in the limit that the bulk stress energy vanishes, because of the modifications to Einstein's equations in the bulk and the nontrivial Israel conditions arising from the Z_2 symmetry. We have shown that this typically leads to difficulties with primordial nucleosynthesis, structure formation, and bounds on the age of the universe.

We have also found that, in a large class of cases, it is impossible to stabilize the radius of the extra dimension (i.e., “pin” the 4-dimensional Planck mass at a fixed value) without fine tuning the equation of state on our brane. The difficulty we find in attempting to stabilize the radius of the extra dimensions to a fixed value is similar to the problem of trying to have a static Einstein universe. We find that the radius of the extra dimension can be frozen at a fixed value in a universe (on the 3-brane) with decreasing energy density only if the equation of state satisfies $w = P/\rho = -2/3$; however, the radius is unstable to slight deviations from this value such that w must be fine-tuned to extreme precision to exactly this number. Although this particular value has recently been discussed, e.g. by Wang *et al* [10] as tentatively favored by overlap of a number of astrophysical observations, the lack of a dynamical mechanism to drive w to this value renders this exact number extremely fine tuned. Even a slowly varying Planck mass, in agreement with observations, requires a tuning. This tuning is a result of the energy density of the universe today being severely mismatched with the energy density set by the scale of the fundamental Planck constant. Although we have only been able to prove the fine tuning required for radius stabilization for specific metrics in the Hořava-Witten scenario (those with conformally flat metric in the $\tau - u$ subspace), we suspect that such a result may generalize to any attempt to stabilize the radius in the context of an extra dimension in which gravitons propagate.

We repeat, however, that this constraint involving the equation of state can be evaded if one assumes that the energy density confined to the 3-brane, ρ , is dominated by the cosmological constant. In that case, the equation of state as well as the energy density can remain constant in time. A cosmological constant which dominates the energy density ρ confined to the brane does not necessarily lead to an inflationary universe; in principle there can be other (nonlocal) contributions from off the brane canceling the effects of the cosmological constant, such that the evolution of our universe today can proceed as usual. The ordinary matter in this case would contribute only negligibly to ρ .

We can apply the results of our paper to the model of Randall and Sundrum [17], which falls into the class of models we have considered here. First, our result that FRW cosmology does not generically result on our brane would apply in their model also. Second, they presumably avoid the problems of stabilizing the orbifold radius by having a cosmological constant dominate the energy density of our brane, as described in the previous paragraph.

Finally, we made some remarks about causality and the horizon problem in theories with extra dimensions. We discussed possible modifications to the standard inflationary picture. For example, the time variation of the gravitational coupling due the changing size of the compactified dimension can be used to solve the horizon problem or at least modify the number of e-foldings needed to solve the horizon problem within the context of inflation. In addition, the modifications indicated here to the standard FRW relation $H^2 \propto \rho$ would change the requirements of inflationary models.

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